

**$k$ -PARAMETER GEODESIC VARIATIONS**

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**ABSTRACT.** Suppose  $S$  is a semispray on a manifold  $M$ . We know that the complete lift  $S^c$  of  $S$  is a semispray on  $TM$  with the property that geodesics of  $S^c$  correspond to Jacobi fields of  $S$ . In this note we generalize this result and show how geodesic variations of  $k$ -variables are related to geodesics of the  $k$ th iterated complete lift of  $S$ . Moreover, for sprays (that is, homogeneous semisprays) we show how geodesic variations of  $(n - 1)$ -variables are related to a natural generalisation of Jacobi tensors.

## 1. INTRODUCTION

Suppose  $S$  is a *semispray* on a smooth manifold  $M$  of dimension  $n$ . Then  $S$  is a vector field on  $TM \setminus \{0\}$  (the manifold on non-zero vectors), and a *geodesic* is a curve  $c: I \rightarrow M$  such that  $c'' = S \circ c'$ . The motivation for studying semisprays is that these provide a minimal mathematical structure for studying curves on  $M$  that solve systems of 2nd order ODEs (ordinary differential equations). In this way, semisprays provide a unified setting for studying geodesics in Riemann, Finsler and Lagrange geometries and for affine and non-linear connections. See for example, [She01a, BM07].

As in Riemann geometry, one can study the variation of geodesics for a semispray, and this leads to a *Jacobi equation* that describes the infinitesimal behaviour of a geodesic variation. More precisely, any geodesic variation induces a Jacobi field, and conversely, any Jacobi field on a compact interval can be represented by a geodesic variation [BM07, BD10a]. The purpose of this paper is to study the analogous representation of geodesic variations of multiple variables. In Riemann geometry we know that geodesic variations of  $n - 1$  parameters are related to Jacobi tensors [EO80]. Here, a *Jacobi tensor* is a  $(1, 1)$ -tensor along a geodesic that satisfies an analogue to the usual Jacobi equation. For similar results in Lorentz geometry, see [EJK98] and [Lar96].

The main results of this paper are Theorem 3.7 and Theorem 4.3. In Theorem 3.7 we derive a Jacobi equation for geodesic variations of  $k \geq 1$  variables in the setting semisprays. The advantage of this result is that it holds for an arbitrary semispray. On the other hand, the disadvantage is that the Jacobi field will be a curve in the  $k$ th iterated tangent bundle and has  $n2^{k-1}$  components. The proof relies on working with iterated complete lifts [Lew00, BD10b]. In Theorem 4.3 we specialise to sprays (that is, homogeneous semisprays) and to manifolds of dimension  $n \geq 2$ . In this setting we show that geodesic variations of  $n - 1$  variables correspond to

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Jacobi tensors on  $M$ . This latter result can be seen as a natural generalisation of the results in [EO80] and [EJK98] described above. The advantage with Jacobi tensors is that they only depend on  $(n-1)^2$  components, and for  $k = n-1$  and  $n \geq 2$  we always have  $(n-1)^2 < n2^{k-1}$ . As an application, we show in Proposition 4.4 how invertible Jacobi tensors can be used to construct global coordinates around a geodesic.

For invertible Jacobi tensors, there is a close relation to *tensor Riccati equations*. We conclude the paper with Section 4.2, which studies this correspondence in more detail. One motivation is that in Riemann geometry the Riccati equation is satisfied by the *shape operator* of hypersurfaces evolving under the geodesic flow [KV86, She01b]. In physics, a motivation for the Riccati equation is that its trace correspond to the *Raychaudhuri equation* used to study the expansion (and collapse) of a family of geodesics [EO80, EK94, JP00]. A related equation is also the complexified Riccati equation which describes the behaviour of amplitude for a propagating wave packet in hyperbolic equations like the wave equation or Maxwell's equations [KKL01, Kac05, Dah08]. Lastly, for time-dependent semisprays, one can define a generalisation of a shape operator for a geodesic vector field, and moreover, show that this shape operator satisfies a tensor Riccati equation [CP84, JP00, JP02]. In Proposition 4.7 we show how this generalised shape operator (for sprays) can be written explicitly using a geodesic variation.

## 2. PRELIMINARIES

We assume that  $M$  is a smooth manifold without boundary and with finite dimension  $n \geq 1$ . By smooth we mean that  $M$  is a topological Hausdorff space with countable base that is locally homeomorphic to  $\mathbb{R}^n$ , and transition maps are  $C^\infty$ -smooth. All objects are assumed to be  $C^\infty$ -smooth on their domains.

For  $r \geq 1$ , let  $T^r M = T \cdots TM$  be the  $r$ :th *iterated tangent bundle*, and for  $r = 0$  let  $T^0 M = M$ . For example, when  $r = 2$  we obtain the second tangent bundle  $TTM$  [Bes78], and in general

$$T^{r+1}M = TT^r M, \quad r \geq 0.$$

For a tangent bundle  $T^{r+1}M$  where  $r \geq 0$ , we denote the canonical projection operator by  $\pi_r: T^{r+1}M \rightarrow T^r M$ . Occasionally we also write  $\pi_{TTM \rightarrow M}$ ,  $\pi_{TM \rightarrow M}, \dots$  instead of  $\pi_0 \circ \pi_1, \pi_0, \dots$ . For  $x \in T^r M$  where  $r \geq 0$  let also  $T_x^{r+1}M = \pi_r^{-1}(x)$ .

We always use canonical local coordinates (induced by local coordinates on  $M$ ) for iterated tangent bundles. If  $x^i$  are local coordinates for  $T^r M$  for some  $r \geq 0$ , we denote induced local coordinates for  $T^{r+1}M$ ,  $T^{r+2}M$ , and  $T^{r+3}M$  by

$$(x, y), \quad (x, y, X, Y), \quad (x, y, X, Y, u, v, U, V).$$

As above, we usually leave out indices for local coordinates and write  $(x, y)$  instead of  $(x^i, y^i)$ .

For  $r \geq 1$ , we treat  $T^r M$  as a vector bundle over the manifold  $T^{r-1}M$  with the vector space structure induced by projection  $\pi_{r-1}: T^r M \rightarrow T^{r-1}M$ . Thus, if  $\{x^i : i = 1, \dots, 2^{r-1}n\}$  are local coordinates for  $T^{r-1}M$ , and  $(x, y)$  are local

coordinates for  $T^r M$ , then vector addition and scalar multiplication are given by

$$\begin{aligned} (1) \quad & (x, y) + (x, \tilde{y}) = (x, y + \tilde{y}), \\ (2) \quad & \lambda \cdot (x, y) = (x, \lambda y). \end{aligned}$$

For  $r \geq 0$ , a *vector field* on an open set  $U \subset T^r M$  is a smooth map  $X: U \rightarrow T^{r+1}M$  such that  $\pi_r \circ X = \text{id}_U$ . The set of all such vector fields is denoted by  $\mathfrak{X}(U)$ .

Suppose that  $\gamma$  is a smooth map  $\gamma: (-\varepsilon, \varepsilon)^k \rightarrow T^r M$  where  $k \geq 1$  and  $r \geq 0$ . Suppose also that  $\gamma(t^1, \dots, t^k) = (x^i(t^1, \dots, t^k))$  in local coordinates for  $T^r M$ . Then the *derivative* of  $\gamma$  with respect to variable  $t^j$  is the curve  $\partial_{t^j} \gamma: (-\varepsilon, \varepsilon)^k \rightarrow T^{r+1}M$  defined by  $\partial_{t^j} \gamma = (x^i, \partial x^i / \partial t^j)$ . When  $k = 1$  we also write  $\gamma' = \partial_t \gamma$  and say that  $\gamma'$  is the *tangent* of  $\gamma$ .

Unless otherwise specified we always assume that  $I$  is an open interval of  $\mathbb{R}$  that contains 0, and we do not exclude unbounded intervals. If  $\phi: M \rightarrow N$  is a smooth map between manifolds, we denote the tangent map  $TM \rightarrow TN$  by  $D\phi$ , and if  $c: I \rightarrow M$  is a curve, then

$$(3) \quad (\phi \circ c)'(t) = D\phi \circ c'(t), \quad t \in I.$$

**Lemma 2.1.** *If  $\xi \in T^r M$  for some  $r \geq 1$  then there exists a map*

$$W: (-\varepsilon, \varepsilon)^r \rightarrow M$$

*such that if  $s^1, \dots, s^r$  are coordinates for  $(-\varepsilon, \varepsilon)^r$  then*

$$\xi = \partial_{s^1} \cdots \partial_{s^r} W(s^1, \dots, s^r)|_{s^1, \dots, s^r=0}.$$

*Proof.* Let  $V = \mathbb{R}^{\dim M}$ . For  $k \geq 1$  let us define functions

$$w^{(k)}: V^{2^k} \times \mathbb{R}^k \rightarrow V$$

as follows. For  $k = 1$  let

$$w^{(1)}(u, v, s^1) = u + s^1 v, \quad u, v \in V, \quad s^1 \in \mathbb{R},$$

and for  $k \geq 2$ , let

$$w^{(k)}(u, v, s^1, \dots, s^k) = w^{(k-1)}(u + s^1 v, s^2, \dots, s^k), \quad \begin{aligned} u, v &\in V^{2^{k-1}}, \\ s^1, \dots, s^k &\in \mathbb{R}. \end{aligned}$$

By induction, it follows that for all  $k \geq 1$  we have

$$\partial_{s^1} \cdots \partial_{s^k} w^{(k)}(u, s^1, \dots, s^k)|_{s^1, \dots, s^k=0} = u$$

for all  $u \in V^{2^k}$  and  $s^1, \dots, s^k \in \mathbb{R}$ . If  $\phi: U \rightarrow V$  is a chart around  $\pi_{T^r M \rightarrow M}(\xi)$ , where  $V$  is as above, then the result follows using equation (3) since the  $r$ th-fold tangent map  $D^{(r)}\phi: T^r U \rightarrow V^{2^r}$  is a coordinate chart around  $\xi$ .  $\square$

**2.1. Canonical involution on  $T^r M$ .** For  $r \geq 2$ , the *canonical involution* is the unique diffeomorphism  $\kappa_r: T^r M \rightarrow T^r M$  that satisfies

$$(4) \quad \partial_t \partial_s c(t, s) = \kappa_r \circ \partial_s \partial_t c(t, s)$$

for all curves  $c: (-\varepsilon, \varepsilon)^2 \rightarrow T^{r-2} M$ . For  $r = 1$ , we define  $\kappa_1 = \text{id}_{TM}$ . For a discussion, see [BD10a] and references therein.

Let  $r \geq 2$ , let  $x^i$  be local coordinates for  $T^{r-2} M$ , and let  $(x, y, X, Y)$  be local coordinates for  $T^r M$ . Then

$$\kappa_r(x, y, X, Y) = (x, X, y, Y).$$

For  $r \geq 1$ , we have identities

$$(5) \quad \kappa_r^2 = \text{id}_{T^r M},$$

$$(6) \quad \pi_r \circ D\kappa_r = \kappa_r \circ \pi_r,$$

$$(7) \quad D\pi_{r-1} = \pi_r \circ \kappa_{r+1},$$

$$(8) \quad D\pi_{r-1} \circ \pi_{r+1} = \pi_r \circ DD\pi_{r-1},$$

$$(9) \quad DD\pi_{r-1} \circ \kappa_{r+2} = \kappa_{r+1} \circ DD\pi_{r-1},$$

$$(10) \quad \pi_{r-1} \circ D\pi_{r-1} = \pi_{r-1} \circ \pi_r.$$

The *slashed tangent bundle* is the open set in  $TM$  defined as

$$TM \setminus \{0\} = \{y \in TM : y \neq 0\}.$$

On  $T^r M$  for  $r \geq 2$  we define *slashed tangent bundles* as open sets

$$T^r M \setminus \{0\} = \{\xi \in T^r M : (D\pi_{T^{r-1}M \rightarrow M})(\xi) \in TM \setminus \{0\}\}.$$

For motivation, see Section 3. Let also  $T^r M \setminus \{0\} = M$  when  $r = 0$ .

**2.2. Iterated lifts for functions.** Next we define the vertical and complete lift of a function  $f: T^r M \rightarrow \mathbb{R}$  on an iterated tangent bundle. When  $r = 0$ , these lifts coincide with the usual vertical and complete lifts defined in [YI73].

**Definition 2.2.** For  $r \geq 0$ , the *vertical lift* of a function  $f \in C^\infty(T^r M \setminus \{0\})$  is the function  $f^v \in C^\infty(T^{r+1} M \setminus \{0\})$  defined by

$$f^v(\xi) = f \circ \pi_r \circ \kappa_{r+1}(\xi), \quad \xi \in T^{r+1} M \setminus \{0\},$$

and the *complete lift* is the function  $f^c \in C^\infty(T^{r+1} M \setminus \{0\})$  defined by

$$f^c(\xi) = df \circ \kappa_{r+1}(\xi), \quad \xi \in T^{r+1} M \setminus \{0\}.$$

Suppose  $f \in C^\infty(T^r M \setminus \{0\})$  where  $r \geq 1$ . If  $x^i$  are local coordinates for  $T^{r-1} M$ , and  $(x, y, X, Y)$  are local coordinates for  $T^{r+1} M$ , then

$$\begin{aligned} f^v(x, y, X, Y) &= f(x, X), \\ f^c(x, y, X, Y) &= \frac{\partial f}{\partial x^a}(x, X)y^a + \frac{\partial f}{\partial y^a}(x, X)Y^a. \end{aligned}$$

### 3. SEMISPRAYS

The motivation for studying semisprays is that they provide a unified framework for studying geodesics for Riemannian metrics, Finsler metrics, non-linear connections, and Lagrange geometries. See [BM07, Sak96, She01a]. Following [BD10a] we next define a semispray on an iterated tangent bundle  $T^r M$ .

**Definition 3.1.** Let  $r \geq 0$ . A *semispray* on  $T^r M$  is a vector field  $S \in \mathfrak{X}(T^{r+1}M \setminus \{0\})$  such that  $(D\pi_r)(S) = \text{id}_{T^{r+1}M \setminus \{0\}}$ .

Let  $S$  be a semispray  $S \in \mathfrak{X}(T^{r+1}M \setminus \{0\})$  for some  $r \geq 0$ . If  $(x, y, X, Y)$  are local coordinates for  $T^{r+2}M$ , then  $S$  is locally of the form

$$(11) \quad \begin{aligned} S(x, y) &= (x^i, y^i, y^i, -2G^i(x, y)) \\ &= y^i \frac{\partial}{\partial x^i} \Big|_{(x, y)} - 2G^i(x, y) \frac{\partial}{\partial y^i} \Big|_{(x, y)}, \end{aligned}$$

where  $G^i$  are functions  $G^i: T^{r+1}U \setminus \{0\} \rightarrow \mathbb{R}$  for some open  $U \subset M$ .

Suppose  $\gamma$  is a curve  $\gamma: I \rightarrow T^r M$  where  $r \geq 0$ . Then we say that  $\gamma$  is *regular* if  $\gamma'(t) \in T^{r+1}M \setminus \{0\}$  for all  $t \in I$ . When  $r = 0$ , this coincides with the usual definition of a regular curve, and when  $r \geq 1$ , curve  $\gamma$  is regular if and only if curve  $\pi_{T^r M \rightarrow M} \circ \gamma: I \rightarrow M$  is regular.

**Definition 3.2.** Suppose  $S$  is a semispray on  $T^r M$  where  $r \geq 0$ . Then a regular curve  $\gamma: I \rightarrow T^r M$  is a *geodesic* of  $S$  if and only if

$$(12) \quad \gamma'' = S \circ \gamma'.$$

Suppose  $S$  is a semispray on  $T^r M$  and locally  $S$  is given by equation (11). Then a regular curve  $\gamma: I \rightarrow T^r M$ ,  $\gamma = (x^i)$ , is a geodesic for  $S$  if and only if

$$(13) \quad \ddot{x}^i + 2G^i \circ \gamma' = 0.$$

In Definition 3.2 we have defined geodesics on open intervals. If  $\gamma$  is a curve on a closed interval we say that  $\gamma$  is a geodesic if  $\gamma$  can be extended into a geodesic defined on an open interval.

A semispray  $S \in \mathfrak{X}(T^{r+1}M \setminus \{0\})$  is a *spray* if  $S$  further satisfies  $[\mathbb{C}_{r+1}, S] = S$ , where  $[\cdot, \cdot]$  is the Lie bracket and  $\mathbb{C}_{r+1}$  is the *Liouville vector field*  $\mathbb{C}_{r+1} \in \mathfrak{X}(T^{r+1}M)$  defined as

$$\mathbb{C}_r(\xi) = \partial_s((1+s)\xi)|_{s=0}, \quad \xi \in T^r M.$$

Then Euler's theorem for homogeneous functions [BCS00] implies that functions  $G^i$  are *positively 2-homogeneous*, that is, if  $(x, y) \in T^{r+1}M \setminus \{0\}$ , then

$$G^i(x, \lambda y) = \lambda^2 G^i(x, y), \quad \lambda > 0.$$

Thus, if  $\gamma$  is a geodesic for a spray  $S$ , the curve  $t \mapsto \gamma(At + B)$  for constants  $A > 0$  and  $B \in \mathbb{R}$  is again a geodesic (when defined).

**3.1. Complete lifts for a semispray.** Suppose  $S$  is a semispray on  $M$ . As described in the introduction, the complete lift of  $S$  is a new semispray  $S^{(1)}$  on  $TM$ . The motivation for studying the complete lift is that geodesics of  $S^{(1)}$  are Jacobi fields of  $S$ , which in turn describe geodesic variations of  $S$ . Next we define *iterated complete lifts* for a semispray  $S$  and Theorem 3.7 will show how these are related to geodesic variations of  $k$  variables.

The below definition for the complete lift can essentially be found in [Lew00, Remark 5.3]. For a further discussion about related lifts, see [BD10a].

**Definition 3.3.** Suppose  $S$  is a semispray on  $M$ . Then the *complete lifts* of  $S$  are semisprays  $S^{(1)}, S^{(2)}, \dots$  on  $TM, TTM, \dots$  defined inductively as follows. For  $r = 0$ , let  $S^{(0)} = S$  and for  $r \geq 0$ , let  $S^{(r+1)}$  be the semispray on  $T^{r+1}M$  defined as

$$S^{(r+1)} = D\kappa_{r+2} \circ \kappa_{r+3} \circ DS^{(r)} \circ \kappa_{r+2}.$$

By induction we see that all  $S^{(0)}, S^{(1)}, \dots$  are semisprays. In fact, if  $S^{(r)}$  is a semispray on  $T^r M$  for some  $r \geq 0$ , and if we write  $S^{(r)}$  as in equation (11), then

$$\begin{aligned} S^{(r+1)} &= \left( x, y, X, Y, X, Y, -2(G^i)^v, -2(G^i)^c \right) \\ (14) \quad &= X^i \frac{\partial}{\partial x^i} + Y^i \frac{\partial}{\partial y^i} - 2(G^i)^v \frac{\partial}{\partial X^i} - 2(G^i)^c \frac{\partial}{\partial Y^i}, \end{aligned}$$

whence  $S^{(r+1)}$  is a semispray on  $T^{r+1}M$ .

**Definition 3.4.** If  $S$  is a semispray on  $M$ , then a *Jacobi field* is a geodesic of  $S^{(1)}$ .

A main motivation for the above definition will be given by Theorem 3.7 below. Alternatively, from equation (14) we see that  $S^{(1)}$  coincides with the usual definition of the complete lift of a semispray on  $M$  [BM07]. Hence, the geodesic equation for  $S^{(1)}$  coincide with the usual Jacobi equation for  $S$  in Riemann, Finsler, or Lagrange geometry [BD10a].

The *geodesic flow* of a semispray  $S^{(r)}$  on  $T^r M$  for  $r \geq 0$  is defined as the flow of  $S^{(r)}$  as a vector field. The next proposition shows how the geodesic flows of  $S^{(0)}, S^{(1)}, \dots$  are related to each other [BD10a]. In particular, if  $S$  is complete (as a vector field), then all complete lifts  $S^{(1)}, S^{(2)}, \dots$  are complete [YI73]. Let us also note that if  $S$  is a spray, then all complete lifts are sprays.

**Proposition 3.5.** Suppose  $S$  is a semispray on  $M$  and  $S^{(0)}, S^{(1)}, S^{(2)}, \dots$  are as above. Suppose furthermore that for each  $r \geq 0$ ,

$$\phi^{(r)} : \mathcal{D}(S^{(r)}) \rightarrow T^{r+1}M \setminus \{0\},$$

is the geodesic flows of semispray  $S^{(r)}$  with maximal domain

$$\mathcal{D}(S^{(r)}) \subset T^{r+1}M \setminus \{0\} \times \mathbb{R}.$$

For all  $r \geq 0$  we then have

$$(15) \quad ((D\pi_r) \times \text{id}_{\mathbb{R}}) \mathcal{D}(S^{(r+1)}) = \mathcal{D}(S^{(r)})$$

and

$$(16) \quad \phi_t^{(r+1)}(\xi) = \kappa_{r+2} \circ D\phi_t^{(r)} \circ \kappa_{r+2}(\xi), \quad (\xi, t) \in \mathcal{D}(S^{(r+1)}),$$

where  $D\phi_t^{(r)}$  is the tangent map of the map  $\xi \mapsto \phi_t^{(r)}(\xi)$  for a fixed  $t$ .

**3.2.  $k$ -parameter geodesic variations.** When  $k = 1$  the next definition reduces to the usual definition of a geodesic variation.

**Definition 3.6.** Let  $k \geq 1$  and let  $c: I \rightarrow M$  be a geodesic for a semispray  $S$  on  $M$ . Then a  $k$ -parameter geodesic variation of  $c$  is a map  $V: I \times (-\varepsilon, \varepsilon)^k \rightarrow M$  such that

- (i)  $V(t, 0, \dots, 0) = c(t)$  for all  $t \in I$ .
- (ii)  $t \mapsto V(t, s^1, \dots, s^k)$  is a geodesic for all  $(s^1, \dots, s^k) \in (-\varepsilon, \varepsilon)^k$ .

**Theorem 3.7.** Let  $S$  be a semispray on  $M$  and let  $r \geq 1$ .

- (i) If  $V: I \times (-\varepsilon, \varepsilon)^k \rightarrow M$  is a  $k$ -parameter geodesic variation for some  $k \geq 1$ , then the curve  $j: I \rightarrow T^r M$  defined as

$$(17) \quad j = \partial_{s^{i_1}} \cdots \partial_{s^{i_r}} V|_{s^1, \dots, s^k=0}$$

is a geodesic of  $S^{(r)}$ . Here  $s^1, \dots, s^k$  are Cartesian coordinates for  $(-\varepsilon, \varepsilon)^k$  and  $i_1, \dots, i_r$  are indices for these coordinates so that  $1 \leq i_1, \dots, i_r \leq k$ .

- (ii) If  $I$  is compact and  $j: I \rightarrow T^r M$  is a geodesic of  $S^{(r)}$ , then  $j$  can be written as

$$(18) \quad j = \partial_{s^1} \cdots \partial_{s^r} V|_{s^1, \dots, s^k=0}, \quad t \in I,$$

for some  $r$ -parameter geodesic variation  $V: I^* \times (-\varepsilon, \varepsilon)^r \rightarrow M$ , where  $I^*$  is an open subset with  $I \subset I^*$ .

*Proof.* For part (i), let  $j^{(1)}, \dots, j^{(r)}$  be maps

$$j^{(p)}: I \times (-\varepsilon, \varepsilon)^k \rightarrow T^p M, \quad p = 1, \dots, r$$

defined as

$$j^{(1)} = \partial_{s^{i_r}} V, \quad j^{(2)} = \partial_{s^{i_{r-1}}} \partial_{s^{i_r}} V, \quad \dots, \quad j^{(r)} = \partial_{s^{i_1}} \cdots \partial_{s^{i_r}} V.$$

By induction we next show that for all  $p = 1, \dots, r$ ,

$$(19) \quad \partial_t^2 j^{(p)} = S^{(p)} \circ \partial_t j^{(p)} \quad \text{on } I \times (-\varepsilon, \varepsilon)^k.$$

For  $p = 1$ , equations (3), (4), (5) and geodesic equation  $\partial_t^2 V = S(\partial_t V)$  yield

$$\begin{aligned} S^{(1)}(\partial_t j^{(1)}) &= D\kappa_2 \circ \kappa_3 \circ DS \circ \kappa_2 \circ \partial_t \partial_{s^{i_r}} V \\ &= \partial_t^2 j^{(1)}. \end{aligned}$$

For  $p \in \{1, \dots, r-1\}$ , the induction step follows by writing  $j^{(p+1)} = \partial_{s^{i_{r-p}}} j^{(p)}$  and repeating the above calculation. Part (i) follows.

For part (ii), Lemma 2.1 implies that there exists a map  $W: (-\delta, \delta)^{r+1} \rightarrow M$  with

$$j'(0) = \partial_{s^0} \cdots \partial_{s^r} W|_{s^0, \dots, s^r=0}$$

With notation as in Proposition 3.5 we obtain

$$\begin{aligned} j(t) &= \pi_r \circ \phi_t^{(r)} \circ j'(0) \\ (20) \quad &= \pi_r \circ \phi_t^{(r)} (\partial_{s^0} \cdots \partial_{s^r} W|_{s^0, \dots, s^r=0}), \quad t \in I. \end{aligned}$$

We know that  $\mathcal{D}(S^{(r)})$  is open [AM78]. For each  $t \in I$ , we can therefore extend the domain of  $\phi_t^{(r)} \partial_{s^0} \cdots \partial_{s^r} W(s^0, \dots, s^r)$  to all  $(t, s^0, \dots, s^r) \in I_t \times (-\delta_t, \delta_t)^{r+1}$  for some open interval  $I_t \ni t$  and  $\delta_t > 0$ . Since  $I$  is compact, we can extend  $I$  into an open interval  $I^* \supset I$  and find an  $\varepsilon^* > 0$  such that  $\phi_t^{(r)} \partial_{s^0} \cdots \partial_{s^r} W$  is defined on  $I^* \times (-\varepsilon^*, \varepsilon^*)^{r+1}$ . By equation (15) it follows that for all  $k \in \{0, \dots, r\}$  we have

$$(\partial_{s^0} \partial_{s^{k+1}} \cdots \partial_{s^r} W, t) \in \mathcal{D}(S^{(r-k)}), \quad (t, s_1, \dots, s_r) \in I^* \times (-\varepsilon^*, \varepsilon^*)^r$$

with convention  $\partial_{s^{k+1}} \cdots \partial_{s^r} W = W$  for  $k = r$ . For  $k \in \{0, \dots, r\}$ , let

$$j^{(k)}: I^* \times (-\varepsilon^*, \varepsilon^*)^{r+1} \rightarrow T^r M$$

be the map defined as

$$j^{(k)} = \partial_{s^1} \cdots \partial_{s^k} (\pi_{r-k} \circ \phi_t^{(r-k)} \circ \partial_{s^0} \partial_{s^{k+1}} \cdots \partial_{s^r} W(s^0, s^1, \dots, s^r)).$$

Equations (3), (4), (7) and (16) imply  $j^{(0)} = \cdots = j^{(r)}$ . Setting  $s^0 = \cdots = s^r = 0$  in equality  $j^{(0)} = j^{(r)}$  and using equation (20) gives

$$j(t) = (\partial_{s^1} \cdots \partial_{s^r} V(t, s^1, \dots, s^r))|_{s^1, \dots, s^r=0}, \quad t \in I,$$

where  $V: I^* \times (-\varepsilon^*, \varepsilon^*)^r \rightarrow M$  is the geodesic variation

$$V(t, s^1, \dots, s^r) = \pi_0 \circ \phi_t^{(0)} \circ \partial_{s^0} W(s^0, s^1, \dots, s^r)|_{s^0=0}.$$

Part (ii) follows.  $\square$

**3.3. Geodesics of  $S^{(r)}$  and Jacobi fields of  $S$ .** The next two propositions describe how geodesics of  $S^{(r)}$  for  $r \geq 2$  are related to geodesics of  $S^{(0)}, \dots, S^{(r-1)}$ . In particular, Proposition 3.8 shows that geodesics of a semispray  $S^{(r)}$  contain geodesics of all lower order lifts  $S^{(0)}, S^{(1)}, \dots, S^{(r)}$ . Thus, by equation (12), we can recover  $S$  from any  $S^{(r)}$  with  $r \geq 1$ .

**Proposition 3.8.** *Suppose  $S$  is a semispray and  $j: I \rightarrow T^r M$  is a geodesic of  $S^{(r)}$ .*

- (i) *If  $r \geq 1$ , then  $\pi_{r-1} \circ j: I \rightarrow T^{r-1} M$  is a geodesic of  $S^{(r-1)}$ .*
- (ii) *If  $r \geq 2$ , then  $\kappa_r \circ j: I \rightarrow T^r M$  is a geodesic of  $S^{(r)}$ .*
- (iii) *If  $r \geq 2$ , then  $D\pi_{r-2} \circ j: I \rightarrow T^{r-1} M$  is a geodesic of  $S^{(r-1)}$ .*

*Proof.* Since all claims are local we may assume that  $I$  is compact. Parts (i) and (ii) follow by Proposition 3.7. Part (iii) follows by equation (7).  $\square$

The next proposition shows that every geodesic of  $S^{(r)}$  induces  $r$  Jacobi fields for  $S$ . The proof of Proposition 3.9 follows by Proposition A.1 in Appendix A.

**Proposition 3.9.** *If  $S$  is a semispray on a manifold  $M$ , and  $j: I \rightarrow T^r M$  is a geodesic of  $S^{(r)}$  where  $r \geq 1$ . Then there are  $r$  distinct maps*

$$p_1^{(r)}, \dots, p_r^{(r)}: T^r M \rightarrow TM$$

*such that*

$$p_1^{(r)} \circ j, \dots, p_r^{(r)} \circ j: I \rightarrow TM$$

*are geodesics of  $S^{(1)}$  (that is, Jacobi fields of  $S$ ).*



Let us also note that if  $S$  is a semispray on  $M$  and  $r \geq 0$ , then a geodesic  $j: I \rightarrow T^r M$  for  $S^{(r)}$  is uniquely determined by  $j'(t_0) \in T^{r+1} M \setminus \{0\}$  for any  $t_0 \in I$ . In contrast, Jacobi fields  $p_1^{(r)} \circ j, \dots, p_r^{(r)} \circ j$  in the above proposition do not determine  $j$ . For example, suppose  $j = (x, y, X, Y)$  is a geodesic of  $S^{(2)}$ , where  $S$  is the flat spray on  $M = \mathbb{R}$ . Then  $x, y, X, Y$  are independent and the geodesic equation reads

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{X} = 0, \quad \ddot{Y} = 0.$$

Now  $p_1^{(2)} \circ j = (x, y)$  and  $p_2^{(2)} \circ j = (x, X)$ , but these do not determine  $Y$ .

#### 4. JACOBI TENSORS AND GEODESIC VARIATIONS

In the previous section, the main result was Theorem 3.7. For a semispray, this theorem shows how geodesics of the  $k$ th iterated complete lift are related to geodesic variations of  $k$ -parameters. Next we specialise this result to sprays, that is, to semisprays that are homogeneous, and to manifolds of dimensions  $n \geq 2$ . The main result in this section is Theorem 4.3, which shows how geodesic variations of  $(n-1)$ -parameters are related to *Jacobi tensors* (equation (22)). Under additional assumptions, this correspondence is known. For the Riemann case, see [EO80] and for the Lorentz case, see [EJK98]. See also [Lar96].

Suppose  $S$  is a semispray on a manifold  $M$ . We know that  $S$  induces a canonical *dynamical covariant derivative* that operates on arbitrary tensors on  $TM \setminus \{0\}$  [BD09]. We will here only need this derivative for tensors along a geodesic [BCD11, Section 3.2] and for a similar operator, see [JP02, Definition 3.3]. Suppose  $c: I \rightarrow M$  is a geodesic for  $S$  and  $X$  is a  $(1, 0)$ -tensor along  $c$ . If locally  $X = X^i(t) \frac{\partial}{\partial x^i} |_{c(t)}$  we define

$$\nabla X = \left( \frac{dX^i}{dt} + N_j^i(c') X^j \right) \frac{\partial}{\partial x^i} \Big|_{c(t)},$$

where  $N_j^i(y) = \frac{\partial G^i}{\partial y^j}(y)$  for  $y \in TM \setminus \{0\}$  and semispray  $S$  is written as in equation (11). Similarly, for a  $(0, 1)$ -tensor  $\alpha = \alpha_i(t) dx^i |_{c(t)}$  we define

$$\nabla \alpha = \left( \frac{d\alpha_i}{dt} - N_i^j(c') \alpha_j \right) dx^i \Big|_{c(t)}.$$

For a function  $f: I \rightarrow M$  along  $c$  we define  $\nabla f = \frac{df}{dt}$ . By the Leibnitz rule, the dynamical covariant derivative  $\nabla$  then extends to tensors of any rank along  $c$  [BD09]. For example, if  $J$  is a  $(1, 1)$ -tensor along  $c$  and  $v$  is a  $(1, 0)$ -tensor, then

$$(21) \quad \nabla(J \circ v) = (\nabla J) \circ v + J \circ \nabla v.$$

We will say that a tensor  $T$  along  $c$  is *parallel* if  $\nabla T = 0$ .

For a semispray, the *Jacobi endomorphism* is a  $(1, 1)$ -tensor on  $TM \setminus \{0\}$ . See [BD09] and references therein. By restricting to  $c'$  we define the *Jacobi endomorphism* as the  $(1, 1)$ -tensor  $\Phi$  along  $c$  defined as  $\Phi(t) = \Phi_j^i(c') \frac{\partial}{\partial x^i} \otimes dx^j |_{c(t)}$ , where

$$\Phi_j^i = \left( 2 \frac{\partial G^i}{\partial x^j} - S \left( \frac{\partial G^i}{\partial y^j} \right) - \frac{\partial G^i}{\partial y^r} \frac{\partial G^r}{\partial y^j} \right)_{c'(t)}.$$

If  $S$  is a spray, then a geodesic  $c: I \rightarrow M$  satisfies  $\nabla c' = 0$  and  $\Phi(c') = 0$ .

**Definition 4.1.** Suppose  $c: I \rightarrow M$  is a geodesic for a semispray. Then a *Jacobi tensor* along  $c$  is a  $(1, 1)$ -tensor  $J$  along  $c$  such that

$$(22) \quad \nabla^2 J + \Phi \circ J = 0.$$

As in the Riemannian case, Jacobi tensors and Jacobi fields are related [EO80]: If  $c: I \rightarrow M$  is a geodesic for a semispray, a  $(1, 1)$ -tensor  $J$  along  $c$  is a Jacobi tensor if and only if  $J \circ v: I \rightarrow TM$  is a Jacobi field for any parallel vector field  $v$  along  $c$ . See [BCD11, Proposition 2.10].

When studying Jacobi tensors in the Riemann setting one usually restrict them to tensors in the normal bundle  $\{c'\}^\perp \rightarrow \{c'\}^\perp$  and such Jacobi tensors can be characterised by their initial values. Our next goal is to prove Proposition 4.2, which shows that a similar result is true also for sprays.

Suppose  $c: I \rightarrow M$  is a geodesic for a spray. Suppose also that  $W$  is an  $(n - 1)$ -dimensional subspace of  $T_{c(0)}M$  and  $c'(0) \notin W$ . We know that parallel transport is a linear isomorphism between tangent spaces. Thus, by parallel transport we can extend any basis  $\{e_i\}_{i=1}^{n-1}$  for  $W$  into linearly independent vectors  $\{e_i(t)\}_{i=1}^{n-1}$  in  $T_{c(t)}M$  for any  $t \in I$ . For  $t \in I$ , let

$$(23) \quad W_t = \text{span}\{e_1(t), \dots, e_{n-1}(t)\}.$$

Then  $W_t$  does not depend on the choice of  $\{e_i\}_{i=1}^{n-1}$ , and  $T_{c(t)}M = \text{span}\{c'(t)\} \oplus W_t$ . Moreover, any vector field  $v: I \rightarrow TM$  along  $c$  can be written as

$$(24) \quad v(t) = v^0(t)c'(t) + \sum_{i=1}^{n-1} v^i(t)e_i(t), \quad t \in I$$

for some smooth functions  $v^0, v^1, \dots, v^{n-1}: I \rightarrow \mathbb{R}$ .

Suppose  $J$  is a  $(1, 1)$ -tensor along a geodesic  $c: I \rightarrow M$  for a spray. Moreover, suppose that  $W \subset T_{c(0)}M$  is a subspace such that  $c'(0) \notin W$  and extensions  $\{W_t : t \in I\}$  are defined as above. Then we say that  $J$  is a *transversal tensor with respect to  $W_t$*  if  $J \circ c'(t) = 0$  and  $\text{Im } J(t) \subset W_t$  for all  $t \in I$ . Equations (21) and (24) imply that  $\nabla J$  is transversal with respect to  $W_t$  if  $J$  is transversal with respect to  $W_t$ .

**Proposition 4.2.** Suppose  $c: I \rightarrow M$  is a geodesic for a spray and  $J$  is a Jacobi tensor along  $c$ . Furthermore, suppose  $W$  is an  $(n - 1)$ -dimensional subspace of  $T_{c(0)}M$  with  $c'(0) \notin W$ . If

- (i)  $\Phi$  is transversal with respect to  $W_t$  and
- (ii)  $J \circ c'(0) = 0$ ,  $(\nabla J) \circ c'(0) = 0$ ,  $\text{Im } J(0) \subset W$ ,  $\text{Im}(\nabla J)(0) \subset W$

then  $J$  is transversal with respect to  $W_t$ .

*Proof.* Let  $j: I \rightarrow TM$  be the vector field  $j = J \circ c'$ . Then  $j$  is a Jacobi field with  $j(0) = \nabla j(0) = 0$ . Thus  $j = 0$ . To complete the proof we need to show that if  $v: I \rightarrow TM$  is a parallel vector field along  $c$ , then  $J \circ v \in W_t$  for all  $t \in I$ . This follows by writing  $j = J \circ v$  as in equation (24) and computing the  $c'(t)$ -component of the Jacobi equation  $\nabla^2 j + \Phi \circ j = 0$ .  $\square$

The next theorem shows how Jacobi tensors and geodesic variations are related in the setting of sprays. For semisprays the analogous result is Theorem 3.7. When  $n \geq 2$ , the correspondence is not unique in neither theorem.

**Theorem 4.3.** *Suppose  $S$  is a spray on a manifold  $M$  of dimension  $n \geq 2$ .*

(i) *Suppose  $V: I \times (-\varepsilon, \varepsilon)^{n-1} \rightarrow M$  is a geodesic variation of  $(n-1)$  parameters and  $(t, s^1, \dots, s^{n-1})$  are coordinates for the domain. Furthermore, suppose  $W$  is an  $(n-1)$ -dimensional vector space in  $T_{c(0)}M$ ,  $c'(0) \notin W$  and  $e_i$  are as in equation (23). Then conditions*

$$(25) \quad J \circ c'(t) = 0,$$

$$(26) \quad J \circ e_a(t) = (\partial_{s^a} V)(t, 0, \dots, 0), \quad a \in \{1, \dots, n-1\}, \quad t \in I.$$

*define a Jacobi tensor  $J$  along  $c$ . Here  $c(t) = V(t, 0, \dots, 0)$ .*

(ii) *Suppose  $c: I \rightarrow M$  is a geodesic, where  $I$  is compact. Furthermore, suppose  $J$  is a transversal Jacobi tensor with respect to  $W_t$  where  $W_t$  is as in equation (23) for some parallel vector fields  $\{e_i(t)\}_{i=1}^{n-1}$ . Then there exists an open interval  $I^* \supset I$  and a geodesic variation  $V: I^* \times (-\varepsilon, \varepsilon)^{n-1} \rightarrow M$  such that equations (25)–(26) hold.*

*Proof.* For part (i), conditions (25)–(26) define a smooth  $(1, 1)$ -tensor  $J$  along  $c$ , and  $J$  is a Jacobi tensor by Theorem 3.7 (i) and the observation after Definition 4.1.

For part (ii), let  $J_a: I \rightarrow TM$  be Jacobi fields  $J_a = J(e_a)$  for  $a \in \{1, \dots, n-1\}$ . If  $(x^i)_{i=1}^n$  are local coordinates around  $c(0)$  we can write  $c(t) = (c^i(t))$  and  $J_a(t) = J_a^i(t) \frac{\partial}{\partial x^i} |_{c(t)}$  for small  $t$ . In these coordinates, let  $U: I \times (-\varepsilon, \varepsilon)^{n-1} \rightarrow M$  be the map

$$U(t, s^1, \dots, s^{n-1}) = \left( c^i(0) + t\dot{c}^i(0) + \sum_{a=1}^{n-1} J_a^i(0)s^a + \sum_{a=1}^{n-1} \dot{J}_a^i(0)ts^a \right)_{i=1}^n.$$

For all  $a \in \{1, \dots, n-1\}$  it follows that

$$(\partial_t J_a)(0) = (\partial_t \partial_{s^a} U)(0, 0, \dots, 0).$$

By a similar compactness argument as in the proof of Theorem 3.7 (ii), there exists an open interval  $I^* \supset I$  and an  $\varepsilon > 0$  such that the map

$$V(t, s^1, \dots, s^{n-1}) = \pi_0 \circ \phi_t^{(0)} \circ (\partial_t U)(0, s^1, \dots, s^{n-1}).$$

defines a geodesic variation  $V: I^* \times (-\varepsilon, \varepsilon)^{n-1} \rightarrow M$ . Then

$$\begin{aligned} J_a(t) &= \pi_1 \circ \phi_t^{(1)} \circ (\partial_t J_a)(0) \\ &= \pi_1 \circ \phi_t^{(1)} \circ (\partial_t \partial_{s^a} V)(0, 0, \dots, 0) \\ &= (\partial_{s^a} V)(t, 0, \dots, 0), \quad a \in \{1, \dots, n-1\}, \quad t \in I \end{aligned}$$

and part (ii) follows.  $\square$

**4.1. Invertible Jacobi tensors.** Suppose  $c$  is a geodesic  $c: I \rightarrow M$  for a spray and  $J$  is a  $(1, 1)$ -tensor along  $c$ . Suppose also that  $J$  is transversal with respect to  $W_t$  for some vector subspaces  $\{W_t : t \in I\}$  as in equation (23). Then we say that  $J$  is *invertible* if  $J|_{W_t}: W_t \rightarrow W_t$  is an invertible linear map for each  $t \in I$ . By  $J^{-1}$  we then denote the transversal  $(1, 1)$ -tensor determined by  $(J^{-1})|_{W_t} = (J|_{W_t})^{-1}$ .

For a semispray  $S$  we say that points  $p, q \in M$  are *conjugate points* if there exists a Jacobi field  $j: I \rightarrow TM$  along a geodesic that connects  $p$  and  $q$ , Jacobi field  $j$  vanishes at  $p$  and  $q$ , but  $j$  is not identically zero. Suppose  $S$  has no conjugate points, and  $J$  is a Jacobi tensor defined on  $I \subset \mathbb{R}$  such that  $J(0) = 0$  and  $\nabla J(0) = \text{Id}$ . Then  $J$  is invertible on  $I \setminus \{0\}$ . See [EK94]. In the Riemann case, it also holds that Jacobi tensors that in addition are *Lagrange tensors* are invertible except at isolated points. See [EO80], and for a discussion of related results in the Lorentz setting, see [EK94].

Proposition 4.4 below shows how the existence of an invertible Jacobi tensor along a geodesic  $c: I \rightarrow M$  implies that geodesics near  $c(I)$  can be straightened out and used to define local coordinates around  $c(I)$ . This gives sufficient conditions when a geodesic variations is a diffeomorphism onto its range. For a similar result for vector fields, see [AM78, Theorem 2.1.9].

**Proposition 4.4.** *Suppose  $c: I \rightarrow M$  is a geodesic for a spray  $S$  on a manifold  $M$  with dimension  $n \geq 2$ , and suppose that  $I$  is compact and  $c$  does not intersect itself. Furthermore, suppose  $J$  is an invertible Jacobi tensor along  $c$ . Then there exists an open interval  $I^* \supset I$ , an  $\varepsilon > 0$  and a map  $V: I^* \times (-\varepsilon, \varepsilon)^{n-1} \rightarrow M$  such that*

- (i)  $V(t, 0, \dots, 0) = c(t)$  for  $t \in I$ ,
- (ii)  $V$  is a diffeomorphism onto its range, that is,  $I^* \times (-\varepsilon, \varepsilon)^{n-1}$  are local coordinates around  $c(I)$ ,
- (iii) for  $(s^1, \dots, s^{n-1}) \in (-\varepsilon, \varepsilon)^{n-1}$  the curve

$$t \mapsto V(t, s^1, \dots, s^{n-1}), \quad t \in I^*$$

*is a geodesic.*

*Proof.* Let  $\{e_i(t)\}_{i=1}^{n-1}$  be a parallel basis such that  $J$  is transversal with respect to  $\text{span}\{e_i(t)\}_{i=1}^{n-1}$ . Then Theorem 4.3 (ii) implies that there exists an open interval  $I^* \supset I$ , an  $\varepsilon > 0$  and a geodesic variation  $V: I^* \times (-\varepsilon, \varepsilon)^{n-1} \rightarrow M$  such that  $V(t, 0, \dots, 0) = c(t)$  for  $t \in I$  and

$$\{\partial_t V, \partial_{s^1} V, \dots, \partial_{s^{n-1}} V\}$$

are linearly independent for  $t \in I$  and  $s^1, \dots, s^{n-1} = 0$ . By the inverse function theorem, we can restrict  $I^* \supset I$  and  $\varepsilon > 0$  such that  $V$  is a local diffeomorphism. Since  $c$  does not intersect itself, we can further apply [Spi79, Lemma 9.19] and restrict  $I^* \supset I$  and  $\varepsilon > 0$  so that  $V$  is a bijection (and hence a diffeomorphism) onto its range.  $\square$

Suppose  $S$  is a spray on  $M$ . Then  $S$  is a *Berwald spray* (or *affine spray*) if in coordinates  $(x^i, y^i)$  for  $TM \setminus \{0\}$ , Christoffel symbols  $\Gamma_{jk}^i = \frac{\partial^2 G^i}{\partial y^j \partial y^k}$  do not depend on  $y^i$ . This condition does not depend on coordinates [She01a, Section 6.1].

If  $S$  is a Berwald spray in Proposition 4.4, then part (iii) implies that  $\Gamma_{11}^i = 0$ . That is, in the terminology of [MV10], Proposition 4.4 gives sufficient conditions for the existence of *pre-semigeodesic coordinates* around a geodesic. Let us note that a similar coordinate system is *Fermi coordinates* around a geodesic, where all Christoffel symbols satisfy  $\Gamma_{jk}^i = 0$ , but only on the geodesic. See [MM63] and for the setting of Berwald sprays, see [Hic65, p. 133] and [Eis27, p. 64].

**4.2. Riccati equation.** Suppose  $J$  is a Jacobi tensor along a geodesic  $c$  for a spray. If  $J$  is invertible (and hence transversal), then  $L = \nabla J \circ J^{-1}$  is another transversal tensor along  $c$  and

$$(27) \quad \nabla L + L^2 + \Phi = 0.$$

This is the *Riccati equation* for a  $(1, 1)$ -tensor, and the above observation demonstrates the relation between the Jacobi tensor equation (22) and the *tensor Riccati equation* (equation (27)). In the Riemann (and Finsler) case, the *shape operator* of metric spheres satisfy the Riccati equation [She01b, Lemma 14.4.2]. For the Lorentz setting, see [EJK98]. The trace of equation (27) is a generalisation of the *Raychaudhuri equation* [EK94, JP00, JP02].

Suppose  $S$  is a semispray on a manifold  $M$ . Then the *connection map* is the map  $K: T(TM \setminus \{0\}) \rightarrow TM$  defined as

$$K(x, y, X, Y) = (x^i, Y^i + N_j^i(y)X^j),$$

where  $S$  is defined as in equation (11) and  $N_j^i = \frac{\partial G^i}{\partial y^j}$ .

**Definition 4.5.** Suppose  $Z$  is a nowhere zero vector field  $Z \in \mathfrak{X}(U)$  defined on an open set  $U \subset M$ . Then  $Z$  defines a  $(1, 1)$ -tensor on  $U$  defined as

$$A_Z = K \circ DZ.$$

If  $Z = Z^i \frac{\partial}{\partial x^i}$  in local coordinates  $(x^i)$  we have

$$A_Z = \left( \frac{\partial Z^i}{\partial x^j} + N_j^i(Z) \right) \frac{\partial}{\partial x^i} \otimes dx^j.$$

From this local expression we see that tensor  $A_Z$  in the above corresponds to tensor  $A_Z$  defined in Definition 4.7 in [JP00]. The next proposition is essentially [JP00, Theorem 6.2] formulated for geodesic variations.

**Proposition 4.6.** Suppose  $c: I \rightarrow M$  is a geodesic for a semispray on a manifold of dimension  $n \geq 2$ , suppose  $V: I \times (-\varepsilon, \varepsilon)^{n-1} \rightarrow M$  is a geodesic variation of  $c$  and  $(t, s^1, \dots, s^{n-1})$  are coordinates for the domain of  $V$ . If  $V$  is a diffeomorphism onto its range, then

$$\nabla A_Z + A_Z^2 + \Phi = 0,$$

where  $Z$  is the vector field induced by  $\frac{\partial}{\partial t}$  and  $A_Z$  is restricted to geodesic  $c$ .

*Proof.* The result follows by a direct computation and using that  $\frac{\partial G^i}{\partial x^j}(c') = 0$  in coordinates  $\{x^i\}_{i=0}^{n-1}$ , where  $x^0 = t$  and  $x^i = s^i$  for  $i = 1, \dots, n-1$ .  $\square$

For sprays and semisprays we know that  $A_Z$  acts as a generalisation of the shape operator [JP00, JP02] when  $Z = \partial_t V$  for a geodesic variation  $V$ . In the setting of sprays, the next proposition gives an explicit expression for  $A_Z$ . In particular, the proposition gives sufficient conditions that imply that  $A_Z = \nabla J \circ J^{-1}$  for a suitable Jacobi tensor. Let us note that in Riemann and Finsler geometry, the shape operator for small geodesic spheres can be written as  $\nabla J \circ J^{-1}$  where  $J$  is a Jacobi tensor with  $J(0) = 0$  and  $\nabla J = \text{Id}$ . See for example [EO80, KV86, She01b]. The next proposition gives sufficient conditions when  $A_Z$  has an analogous representation. For a discussion of equation (29), see [JP00, EO80].

**Proposition 4.7.** *Suppose  $c: I \rightarrow M$  is a geodesic for a spray on a manifold of dimension  $n \geq 2$ , suppose  $V: I \times (-\varepsilon, \varepsilon)^{n-1} \rightarrow M$  is a geodesic variation of  $c$ , suppose  $(t, s^1, \dots, s^{n-1})$  are coordinates for the domain of  $V$ , and suppose  $W$  is an  $(n-1)$ -dimensional subspace in  $T_{c(0)}M$  with  $c'(0) \notin W$ , and  $J$  is a  $(1, 1)$ -tensor along  $c$  defined as in Theorem 4.3 (i). Furthermore, suppose*

- (i) *for some open interval  $I_0 \subset I$ , the restriction  $V: I_0 \times (-\varepsilon, \varepsilon)^{n-1} \rightarrow M$  is a diffeomorphism onto its range.*
- (ii) *on  $I_0$ ,  $J$  is transversal with respect to  $W_t$ .*

*Then  $J$  is invertible on  $I_0$ , and on  $I_0$  we have*

$$(28) \quad A_Z = \nabla J \circ J^{-1},$$

$$(29) \quad \frac{d}{dt}(\det J) = \text{trace } A_Z \cdot \det J,$$

*where  $A_Z$  is the  $(1, 1)$ -tensor along  $c$  associated to vector field  $Z = \partial_t V$  and  $\det J$  is the determinant of the transverse part of  $J$ . In particular,  $A_Z$  is transversal on  $I_0$ .*

*Proof.* Let  $e_1, \dots, e_{n-1}$  be parallel vector fields along  $c$  such that  $J$  is defined by equations (25)–(26) and  $W = \text{span}\{e_a(0)\}_{a=1}^{n-1}$ . Then  $J$  is transversal with respect to  $\text{span}\{e_a(t): I_0 \rightarrow TM\}_{a=1}^{n-1}$ , and assumption (i) implies that  $J$  is invertible on  $I_0$ . Let

$$B = \{J \circ v: I_0 \rightarrow TM : v \text{ is a parallel } (1, 0)\text{-tensor along } c\}.$$

Then  $B$  is a vector space over  $\mathbb{R}$  and  $\dim B = n-1$ . For  $a \in \{1, \dots, n-1\}$  we also have  $(\partial_{s^a} V)(t, 0, \dots, 0) \in B$ . Now  $(t, s^1, \dots, s^{n-1})$  are local coordinates around  $c(I_0)$ , and any  $j \in B$  can be written as

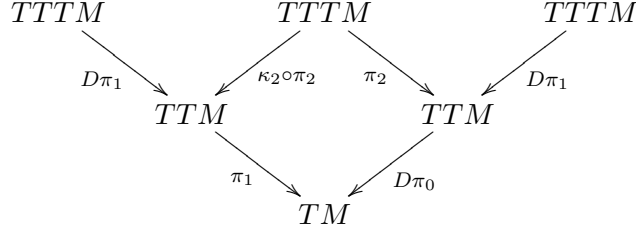
$$j(t) = \sum_{a=1}^{n-1} J^a \frac{\partial}{\partial s^a} \Big|_{c(t)}, \quad t \in I_0$$

for some constants  $J^1, \dots, J^{n-1} \in \mathbb{R}$ . A direct computation shows that  $\nabla j = A_Z \circ j$  on  $I_0$  and equation (28) follows by equation (21). Equation (29) follows by the argument in [EO80, Lemma 1], that is, essentially by *Liouville's formula*.  $\square$

## APPENDIX A. CANONICAL PROJECTIONS $T^r M \rightarrow TM$

It is well known that there are two distinct canonical projections  $TTM \rightarrow TM$ , namely  $\pi_1, D\pi_0: TTM \rightarrow TM$ . See for example [Bes78, BD10a]. Similarly, for

$TTTM$  there are three distinct canonical projections  $TTTM \rightarrow TM$  illustrated in the below commutative diagram:



Next we generalize these results and show that for any  $r \geq 2$  there are  $r$  distinct canonical projections

$$p_1^{(r)}, \dots, p_r^{(r)} : T^r M \rightarrow TM.$$

In this appendix we construct these projections and prove a number of technical properties. For geometric implications of these results, see Section 3.3.

The maps are defined by induction. Let  $p_1^{(1)} : TM \rightarrow TM$  be the identity map

$$p_1^{(1)} = \text{id}_{TM}.$$

For  $r \geq 2$  we define maps  $p_1^{(r)}, \dots, p_r^{(r)} : T^r M \rightarrow TM$  as

$$p_i^{(r)} = \begin{cases} p_i^{(r-1)} \circ \pi_{r-1}, & \text{for } i = 1, \dots, r-1, \\ p_{r-1}^{(r-1)} \circ D\pi_{r-2}, & \text{for } i = r. \end{cases}$$

For example, for  $TM, TTM, TTTM$  we obtain projection maps

$$\begin{aligned}
 p_1^{(1)} &= \text{id}_{TM}, \\
 p_1^{(2)} &= \pi_1, \quad p_2^{(2)} = D\pi_0, \\
 p_1^{(3)} &= \pi_1 \circ \pi_2, \quad p_2^{(3)} = D\pi_0 \circ \pi_2, \quad p_3^{(3)} = D(\pi_{T^2 M \rightarrow M}).
 \end{aligned}$$

**Proposition A.1.** *Let  $r \geq 1$ .*

- (i) *The maps  $p_1^{(r)}, \dots, p_r^{(r)} : T^r M \rightarrow TM$  are distinct.*
- (ii)  *$\pi_0 \circ p_i^{(r)} = \pi_{T^r M \rightarrow M}$  for all  $i = 1, \dots, r$ .*
- (iii)  *$p_1^{(r)} = \pi_{T^r M \rightarrow TM}$  and  $p_r^{(r)} = D(\pi_{T^{r-1} M \rightarrow M})$  for  $r \geq 2$ .*
- (iv) *Suppose  $V : I \times (-\varepsilon, \varepsilon)^k \rightarrow M$  is a map where  $k \geq 1$ . Then*

$$p_a^{(r)} \circ \partial_{s^1} \cdots \partial_{s^r} V = \partial_{s^{r-a+1}} V, \quad a = 1, \dots, r,$$

*where  $s^1, \dots, s^r$  index coordinates for  $(-\varepsilon, \varepsilon)^k$  so that  $1 \leq s^1, \dots, s^r \leq k$ .*

*Proof.* For (i) the claim is clear for  $r = 1, 2, 3$  and for the induction step, suppose that (i) holds for some  $r \geq 3$ . Then

$$(30) \quad p_i^{(r)} \neq p_j^{(r)}, \quad \text{for all } 1 \leq i < j \leq r-1,$$

$$(31) \quad p_i^{(r)} \neq p_r^{(r)}, \quad \text{for all } 1 \leq i \leq r-1.$$

Applying  $\pi_r$  to equation (30) from the right gives that  $\{p_i^{(r+1)}\}_{i=1}^{r-1}$  are distinct. Similarly, applying  $\pi_r$  to equation (31) gives that  $p_i^{(r+1)} \neq p_r^{(r+1)}$  for all  $i = 1, \dots, r-1$ . Applying  $D\pi_{r-1}$  to equation (31) and using equation (10) gives that

$p_i^{(r+1)} \neq p_{r+1}^{(r+1)}$  for all  $i = 1, \dots, r-1$ . Applying  $DD\pi_{r-2}$  to equation (31) with  $i = r-1$  yields

$$p_{r-1}^{(r-1)} \circ \pi_{r-1} \circ DD\pi_{r-2} \neq p_{r-1}^{(r-1)} \circ D\pi_{r-2} \circ DD\pi_{r-2}$$

whence equations (8) and (10) imply that  $p_r^{(r+1)} \neq p_{r+1}^{(r+1)}$ . Part (ii) follows by induction and equation (10). Part (iii) follows directly by induction. For part (iv), let us fix  $k \geq 1$ . The claim is clear for  $r = 1, 2$  and for the induction step suppose that the claim holds for some  $r \geq 2$ . For any  $1 \leq s^1, \dots, s^{r+1} \leq k$  and  $a = 1, \dots, r+1$  we then have

$$\begin{aligned} p_a^{(r+1)} \partial_{s^1} \dots \partial_{s^{r+1}} V &= \begin{cases} p_a^{(r)} \circ \pi_r \circ \partial_{s^1} \dots \partial_{s^{r+1}} V, & a = 1, \dots, r, \\ p_r^{(r)} \circ D\pi_{r-1} \circ \partial_{s^1} \dots \partial_{s^{r+1}} V, & a = r+1 \end{cases} \\ &= \begin{cases} p_a^{(r)} \circ \partial_{s^2} \dots \partial_{s^{r+1}} V, & a = 1, \dots, r, \\ p_r^{(r)} \circ \partial_{s^1} \partial_{s^3} \dots \partial_{s^{r+1}} V, & a = r+1. \end{cases} \end{aligned}$$

Part (iv) follows by using the induction assumption in the upper branch and by using part (iii) in the lower branch.  $\square$

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